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SAINT-VENANT END EFFECTS FOR PLANE DEFORMATION OF SANDWICH STRIPS

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Abstract—The purpose of this paper is to draw attention to the fact that the routine application of Saint-Venant's principle in the solution of elasticity problems for sandwich type structures is not justified in general. This is illustrated in the context of the plane problem of elasticity for a sandwich strip composed of two dissimilar isotropic materials. The exponential decay of end effects is characterized in terms of a complex eigenvalue. For the case of a sandwich with relatively soft middle core, the characteristic decay length is shown to be much greater than that for an homogeneous isotropic strip. The results are analogous to those obtained previously by the authors for highly anisotropic and composite materials.

1. INTRODUCTION

The fundamental role played by Saint-Venant's principle in the applications of elasticity theory can hardly be overemphasized. Thus the simplification arising from consideration of resultant boundary conditions instead of mathematically exact pointwise conditions has facilitated the solution of a whole class of problems of practical interest. It is the purpose of this paper to draw attention to the fact that the routine invocation of Saint-Venant's principle in the solution of problems involving sandwich type structures is not justified in general however. A similar limitation arises in the context of highly anisotropic and composite materials.

The validity of Saint-Venant's principle for anisotropic materials was investigated theoretically by one of the authors some time ago[1–3]. These papers were concerned with assessing the effect of anisotropy on the decay of stresses with distance from the boundary of an elastic solid subjected to self-equilibrated loads. Using methods involving energy-decay inequalities of the type developed by Knowles[4] and Toupin[5], lower bounds (in terms of the elastic constants) were obtained for the rate of exponential decay of stresses, giving rise to upper bounds for "characteristic decay lengths". In [1, 2] the plane problem was treated, while the torsionless axisymmetric problem for a circular cylinder was considered in [3]. The results for the plane problem, when compared with the estimates of Knowles for the isotropic case[4], suggested that the stress decay would be fastest for the isotropic material. In particular, the case of transverse isotropy relevant to fiber-reinforced composites was examined in [2]. For such highly anisotropic materials, a characteristic decay length§ λ of order $b(E/G)^{1/2}$ was predicted , where b is the maximum dimension perpendicular to the fibers and E, G are the longitudinal Young's modulus and shear modulus respectively. When E/G is large, therefore, very slow stress decay is anticipated with end effects being transmitted a considerable distance from the loaded ends.¶ For example, for the case of a graphite/epoxy composite, this ratio has the value 33.3 compared with a range of values of between 2 and 3 for the isotropic case. Thus, for the problem of a rectangular strip loaded only at the short ends, the usual engineering approximation of neglecting Saint-Venant end effects at distances of about one width from the ends is justified in the isotropic case. However, for the graphite/epoxy composite, such approximations are justified only at distances about four times larger. Of course, for even more highly anisotropic materials the effects are more pronounced.

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§Thus λ^{-1} is the exponential decay rate.
¶Such a stress "channelling" phenomenon was observed in the idealized theory for fiber-reinforced composites (see, e.g. [6, 7]). A similar type of "load diffusion" is also considered in a recent paper[8].

Recently this issue has been the subject of extensive experimental investigations[9–11] carried out at the University of Bristol, England. Initial attention to the question at hand was drawn in the course of conducting torsion pendulum tests for measurements of the longitudinal shear modulus for a highly anisotropic polymeric microcomposite[9]. It was found[9] that the calculated values of the shear modulus varied with the specimen aspect ratio (length/width ratio), uniform results being obtained only for samples with aspect ratios of about 100. This is of course in complete contrast with results of testing procedures for isotropic materials. For the block copolymer microcomposite used in[9], E/G is about 280 and so characteristic decay lengths of the order of 16–17 sample widths might be expected on the basis of the theoretical predictions. Further tests are described in[10, 11]. In particular, the results of a simple tension test for the measurement of the longitudinal Young's modulus for a highly drawn polyethylene are described in [10, 11]. For this material, $(E/G)^{1/2} = 15.1$[11] and so the theoretical results would suggest that specimens with aspect ratios of at least 30 would be required for satisfactory measurements. This was found out to be the case[10, 11].

In a recent paper[12], the authors have considered the stress decay for plane deformation of an anisotropic rectangular strip subject to nonzero loading on the short ends only. Using analogues of the celebrated Fadle–Papkovich self-equilibrating eigenfunctions for isotropic elasticity (see, e.g. [13]), the exact decay of end effects can be characterized in terms of a complex eigenvalue with smallest real part (see [13], p. 62). For highly anisotropic transversely isotropic materials slow decay rates were obtained confirming the lower bound estimates obtained in [1, 2] using energy arguments. The present paper is concerned with a similar characterization of the exact decay rates for plane deformation of a sandwich strip as shown in Fig. 1. The identical homogeneous isotropic face materials occupying two layers of equal thickness enclose a dissimilar homogeneous isotropic core. Saint-Venant's principle for such a strip has been investigated experimentally by Alwar[14]. For the case when the Young's modulus of the core is small compared with that for the face layers, it was shown using photoelasticity techniques in [14] that the decay of Saint-Venant end effects is much slower than that for a single strip. We wish to provide a quantitative theoretical analysis for such results here.

In the next section, we derive transcendental equations for the eigenvalues arising in the solution of the biharmonic equation for the sandwich strip, perfectly bonded at the interfaces and traction free on the lateral sides. Our analysis is similar to that of Hess[15, 16] concerned with the end problem for two-layered strips. As was mentioned above, the dominant exponential decay rate can be characterized in terms of a complex eigenvalue with smallest real part. In Section 3, we consider the two limiting cases of a rigid and soft core respectively. The former leads to the case of a strip bonded to a rigid foundation, previously considered in [15]. The latter involves the case of a strip with built in edges treated by Little[17]. Finally, in Section 4, some numerical and asymptotic results are presented and discussed. In particular, for the case

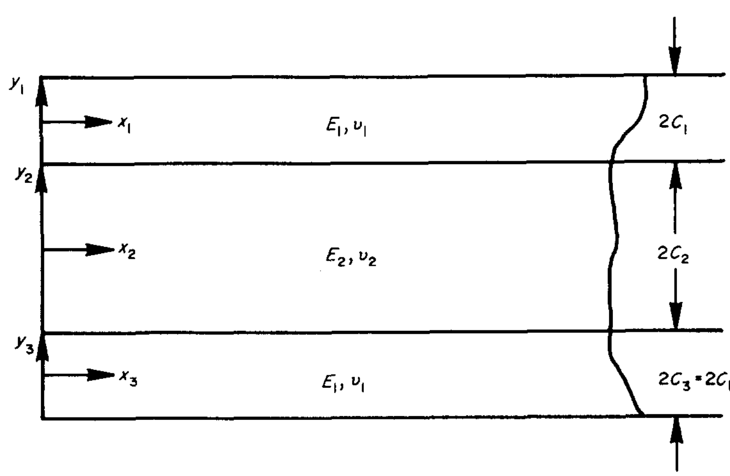

Fig. 1. Semi-infinite sandwich strip.

of a relatively soft inner core, we obtain an asymptotic estimate exhibiting the slow attenuation of end effects.

2. CHARACTERIZATION OF EIGENVALUES

We consider first the case of a semi-infinite homogeneous isotropic elastic strip of width $2c$ in the $x-y$ plane. For problems of plane elasticity, the in-plane stress components are given in terms of the Airy stress function $\varphi(x, y)$ by

$$\sigma_x = \varphi_{yy}, \quad \sigma_y = \varphi_{xx}, \quad \tau_{xy} = -\varphi_{xy}, \tag{2.1}$$

where the subscript notation on φ denotes partial differentiation. Here φ satisfies the biharmonic equation

$$\varphi_{xxxx} + 2\varphi_{xxyy} + \varphi_{yyyy} = 0. \tag{2.2}$$

We seek a solution of (2.2) in the form

$$\varphi(x, y) = e^{-\gamma x/c} F(y), \tag{2.3}$$

where γ is a constant, which may be complex. Substituting into eqn (2.2) yields the fourth-order ordinary differential equation

$$F^{(IV)} + 2\left(\frac{\gamma}{c}\right)^2 F'' + \left(\frac{\gamma}{c}\right)^4 F = 0, \tag{2.4}$$

where the prime denotes differentiation with respect to y. The stresses may then be written as

$$\sigma_x = e^{-\gamma x/c} F'', \quad \sigma_y = \left(\frac{\gamma}{c}\right)^2 e^{-\gamma x/c} F, \quad \tau_{xy} = \frac{\gamma}{c} e^{-\gamma x/c} F'. \tag{2.5}$$

On integrating the appropriate stress–strain relations for plane strain and generalized plane stress respectively, we obtain the displacements $u(x, y)$, $v(x, y)$ (to within a rigid body displacement) as

$$u = \frac{-e^{-\gamma x/c}}{E}\left\{\frac{\alpha c}{\gamma} F'' + \frac{[\alpha - (1+\nu)]\gamma}{c} F\right\}, \tag{2.6}$$

$$v = \frac{-e^{-\gamma x/c}}{E}\left\{\alpha\left(\frac{c}{\gamma}\right)^2 F''' + [\alpha + (1+\nu)] F'\right\}, \tag{2.7}$$

where

$$\left.\begin{aligned} \alpha &= 1-\nu^2 \\ &= 1. \end{aligned}\right\} \begin{aligned} &\text{(Plane Strain)} \\ &\text{(Generalized Plane Stress).} \end{aligned} \tag{2.8}$$

Here E, ν denote the Young's modulus and Poisson ratio respectively.

The ordinary differential eqn (2.4) has the solution

$$F(y) = \left(\frac{c}{\gamma}\right)^2 \left[\left(a_1 + a_2\gamma\frac{y}{c}\right)\cos\gamma\frac{y}{c} + \left(a_3 + a_4\gamma\frac{y}{c}\right)\sin\frac{\gamma y}{c}\right], \tag{2.9}$$

where a_i are constants and the multiplicative factor is introduced for convenience in the calculations to follow. Substituting from (2.9) into (2.5)–(2.7), we may express the stresses and displacements in convenient matrix notation as

$$\begin{pmatrix} \sigma_y \\ \tau_{xy} \\ u \\ v \end{pmatrix} = \frac{c\,e^{-\gamma x/c}}{E\gamma} \begin{pmatrix} \frac{E\gamma a_1}{c} & \frac{E\gamma a_2}{c} & \frac{E\gamma a_3}{c} & \frac{E\gamma a_4}{c} \\ \frac{E\gamma(a_2+a_3)}{c} & \frac{E\gamma a_4}{c} & \frac{E\gamma(a_4-a_1)}{c} & \frac{-E\gamma a_2}{c} \\ (1+\nu)a_1-2\alpha a_4 & (1+\nu)a_2 & 2\alpha a_2+(1+\nu)a_3 & (1+\nu)a_4 \\ [(2\alpha-1-\nu)a_2-(1+\nu)a_3] & -(1+\nu)a_4 & [(1+\nu)a_1+(2\alpha-1-\nu)a_4] & (1+\nu)a_2 \end{pmatrix} \begin{pmatrix} \cos\frac{\gamma y}{c} \\ \frac{\gamma y}{c}\cos\frac{\gamma y}{c} \\ \sin\frac{\gamma y}{c} \\ \frac{\gamma y}{c}\sin\frac{\gamma y}{c} \end{pmatrix}. \tag{2.10}$$

If it is assumed that the lateral surfaces of the strip are traction free, then it is well-known[13] that we obtain an eigenvalue problem for eqn (2.4). The complex eigenvalues γ are characterized as the nonzero roots of the transcendental equations

$$\sin 2\gamma \pm 2\gamma = 0. \tag{2.11}$$

The upper sign in (2.11) gives rise to eigenfunctions $F(y)$ which are even in y and correspond to symmetric deformations, the lower sign leads to odd eigenfunctions corresponding to skew-symmetric deformations. These are the celebrated Fadle–Papkovich eigenfunctions. Analogues of these functions for the case of anisotropic elasticity have been discussed in a recent paper[12] by the authors. The stresses associated with the eigenfunctions F are self-equilibrating and so the representation (2.5) provides a convenient framework for investigating the decay of end effects underlying Saint-Venant's principle. The dominant exponential decay rate is clearly governed by the smallest positive real part of the eigenvalues γ (see [13], p. 62).

We now consider the modifications necessary to adapt the preceding considerations to the case of the sandwich strip shown in Fig. 1. The lateral surfaces of the strip are assumed to be free of traction and we assume perfect bond at the interfaces. Following the scheme developed by Hess[15] for the two-layered strip, we introduce local coordinates (x_i, y_i) in each layer as shown in Fig. 1. A general solution of the form (2.10) may then be written for each of the three layers. At the interfaces, we have the continuity conditions

$$(\sigma_y^{(i)}, \tau_{xy}^{(i)})y_i = -c_i = (\sigma_y^{(i+1)}, \tau_{xy}^{(i+1)})y_{i+1} = c_{i+1} \tag{2.12}$$

$$(u^{(i)}, v^{(i)})y_1 = -c_i = (u^{(i+1)}, v^{(i+1)})y_{i+1} = c_{i+1}, \tag{2.13}$$

for $i = 1, 2$ where $c_3 = c_1$ (see Fig. 1). On using (2.10), (2.12), it can be readily seen that

$$\gamma_3 = \gamma_1, \quad \frac{\gamma_1}{c_1} = \frac{\gamma_2}{c_2}. \tag{2.14}$$

On using the boundary conditions and interface conditions (2.12), (2.13), the transcendental equations analogous to (2.11) can be derived. The analysis is similar to that of Hess in [15] and details may be found in [18]. The equations can be written as

$$\begin{aligned}(E')^2 d_1(\beta_2-2)\{(\beta_2+2)\sin 2\gamma_2 \pm 2(\beta_2-2)\gamma_2\} + 2E'\{2[\sin(4\gamma_1)\cos(2\gamma_2) \pm 4\gamma_1] \\ -[\beta_1\sin^2(2\gamma_1) - 4(\beta_1-2)\gamma_1^2][\beta_2\sin(2\gamma_2) \pm 2(\beta_2-2)\gamma_2]\} \\ +[\sin(2\gamma_2) \pm 2\gamma_2][(\beta_1^2-4)\sin^2(2\gamma_1) - 4(\beta_1-2)^2\gamma_1^2 + 4] = 0,\end{aligned} \tag{2.15}$$

where the upper sign corresponds to symmetric deformation of the middle layer and the lower sign corresponds to skew-symmetric deformation. In (2.15), we have used the notation

$$E' = \frac{E_1\alpha_2}{E_2\alpha_1}, \quad \beta_i = 2 - \frac{(1+\nu_i)}{\alpha_i} \quad (i = 1, 2), \tag{2.16}$$

$$d_1 = \sin^2(2\gamma_1) - (2\gamma_1)^2, \tag{2.17}$$

where α_i $(i = 1, 2)$ are defined in an obvious way by eqn (2.8). After considerable manipulation, eqns (2.15) may be reduced to the following forms

$$8E' \sin(2\gamma_1)\cos 2(\gamma_1+\gamma_2) \pm 16E'\gamma_1 + [\sin(2\gamma_2) \pm 2\gamma_2][4 + K_1K_2d_1 - 16K_1\gamma_1^2] \mp 8K_1E'd_1\gamma_2 = 0, \tag{2.18}$$

where

$$K_{1,2} = E'\beta_2 - \beta_1 \mp 2(E'-1). \tag{2.19}$$

Each of the two eqns (2.18) may be written as a transcendental equation for γ_1 alone, on eliminating γ_2 by $(2.14)_2$. The roots of these equations are located in the four quadrants of the complex plane symmetrically with respect to the real and imaginary axes. Hence, it is sufficient for our purposes to confine attention to roots in the first quadrant.

3. ROOTS OF THE EIGENVALUE EQUATION

To investigate the exponential decay of stresses, we are concerned primarily in the sequel with the root of (2.18) with smallest real part. The dependence of this eigenvalue on the relative thickness of the layers is conveniently described by means of the volume fraction f of the face material defined as the ratio of the width of the face material to the total strip width. Thus

$$f = \frac{4c_1}{4c_1 + 2c_2}, \tag{3.1}$$

and so from $(2.14)_2$ we obtain

$$\gamma_2 = \left(\frac{c_2}{c_1}\right)\gamma_1 = \frac{2(1-f)}{f}\gamma_1. \tag{3.2}$$

To compare the exponential decay rate $\gamma_1/c_1 = \gamma_2/c_2$ with that for a homogeneous strip of the same total width, we set

$$\frac{\gamma}{2c_1 + c_2} = \frac{\gamma_1}{c_1} = \frac{\gamma_2}{c_2}, \tag{3.3}$$

and so from (3.1) we obtain

$$\gamma_1 = \frac{f\gamma}{2}, \quad \gamma_2 = (1-f)\gamma. \tag{3.4}$$

Thus eqns (2.18) may be written as

$$\begin{aligned} &8E' \sin(f\gamma)\cos[(2-f)\gamma] \pm 8E'f\gamma + \{\sin[2(1-f)\gamma] \pm 2(1-f)\gamma\} \\ &\quad \times \{4 + K_1K_2d_1 - 4K_1f^2\gamma^2\} \mp 8K_1E'(1-f)d_1\gamma = 0, \end{aligned} \tag{3.5}$$

where from (2.17), $(3.4)_1$, we have

$$d_1 = \sin^2(f\gamma) - (f\gamma)^2. \tag{3.6}$$

We consider first of all some limiting cases of (3.5) which serve as a guide for the further discussion carried out in Section 4. For the case of a homogeneous strip where $E_1 = E_2$, $\nu_1 = \nu_2$, we find from (2.19) that $K_1 = 0$, $K_2 = 0$ and (3.5) reduces to the transcendental eqn (2.11) for the Fadle–Papkovich eigenvalues. The roots of (2.11) have been tabulated in several places. We refer to [19] for an extensive tabulation. The first even and odd eigenvalues are given by

$$\gamma = 2.106196 + i\,1.125364, \quad \gamma = 3.748838 + i\,1.384339. \tag{3.7}$$

We shall be primarily concerned in what follows with the two limiting cases corresponding to a rigid and soft middle layer respectively.

Case A. $E_2 \gg E_1$ ($E' \to 0$)

In this case, eqns (2.18) reduce to

$$(\beta_1^2 - 4)\sin^2(2\gamma_1) - 4(\beta_1 - 2)^2\gamma_1^2 + 4 = 0. \tag{3.8}$$

This equation is the eigenvalue equation for a homogeneous isotropic elastic strip bonded to a rigid foundation. This may be verified directly by using (2.10) and using appropriate boundary conditions at the lateral sides. Since β_1 defined in (2.16) depends on Poisson's ratio, so will the roots of (3.8). Roots of (3.8) for various values of Poisson's ratio have been tabulated by Hess in [15]. The first nonzero roots (real) for plane strain are given in Table 1 here for various values of Poisson's ratio. These values decrease with increasing value of this elastic constant. On comparing with (3.7), it is seen that the exponential decay of stresses for the strip bonded to a rigid foundation is slower than that for a strip traction-free on both lateral surfaces.† When $\nu = 0.2$, for example, the decay rate in the former case is about four times slower than that for the latter case. Thus while the usual engineering approximation of neglecting Saint–Venant end effects at distances of about one width from the end $x = 0$ is justified for the latter situation, for the former such approximations are justified only at distances about four times larger.

Table 1. The first nonzero roots (real) of eqns (3.8), $(3.9)_1$ for plane strain

Poisson's ratio	Equation (3.8)	Equation $(3.9)_1$
0.00	0.59481	1.13939
0.20	0.50027	1.00018
0.25	0.47818	0.94779
0.30	0.45638	0.88299
0.35	0.43476	0.79967
0.40	0.41315	0.68635
0.45	0.39146	0.51338
0.50	0.36954	2.10620*

*Real part of eigenvalue $(3.7)_1$.

Case B. $E_2 \ll E_1$ ($E' \to \infty$)

In this case, eqns (2.18) reduce to

$$(\beta_2 + 2)\sin(2\gamma_2) \pm 2(\beta_2 - 2)\gamma_2 = 0. \tag{3.9}$$

These transcendental equations were considered by Little [17] in connection with a semi-infinite strip with built-in edges in plane stress. The first nonzero roots (real) of eqn $(3.9)_1$ for plane strain are given in Table 1 here for various values of Poisson's ratio. The eigenvalues in this case are seen to be about twice those for the strip bonded to a rigid foundation, as might be anticipated. It is of interest to observe the discontinuity in the values of γ_2 as ν tends to one half. For nearly incompressible materials, $(3.9)_1$ has a real root close to zero. Using a Taylor expansion in $(3.9)_1$, it can be shown that

$$\gamma_2 \sim \sqrt{\left(\frac{3(1-2\nu_2)}{3-4\nu_2}\right)} \quad \text{as} \quad \nu_2 \to 1/2. \tag{3.10}$$

A slow decay rate for a nearly incompressible built-in strip might be expected. The fully incompressible plane strain equations, however, involving a hydrostatic pressure term, can be simplified by introducing a "stream function" leading to the biharmonic equation. The built-in boundary conditions then lead to (2.11), with first eigenvalue given by $(3.7)_1$, as in the last entry of Table 1.

†Recall that the stresses are independent of the elastic constants in this case, assuming that traction boundary conditions are imposed at the short end.

Equations (3.5) may be solved by iteration starting from the roots of (2.11). For details, we refer to [18]. Table 2 shows the first eigenvalues for plane strain for various values of Poisson's ratio when $E_1/E_2 = 1$ and $f = 1/2$. From the table and the structure of eqns (3.5), it is clear that the mismatch in Poisson ratios has minimal effect on the behavior of the eigenvalues. Thus, for the remainder of this work, we have taken $\nu_1 = \nu_2 = 0.3$ for convenience, and confine our attention to the case of plane strain.

Table 2. The first roots of eqn $(3.5)_1$ for plane strain. ($E_1 = E_2$, $f = 1/2$)

ν_2 \ ν_1	0.0	0.1	0.2	0.3	0.4	0.5
0.0	2.106196 +i1.125364	2.132613 +i1.071271	2.158589 +i1.004050	2.185038 +i0.9193346	2.213174 +i0.8089630	2.244766 +i0.6547567
0.1	2.080084 +i1.177109	2.106196 +i1.125364	2.131545 +i1.061752	2.156987 +i0.9827187	2.183646 +i0.8819154	2.213169 +i0.7463223
0.2	2.054953 +i1.236941	2.081124 +i1.186599	2.106196 +i1.125364	2.130978 +i.050276	2.156514 +i0.9562211	2.184336 +i0.8333258
0.3	2.029872 +i1.308061	2.056530 +i1.258124	2.081714 +i1.198067	2.106196 +i1.125364	2.130948 +i1.035782	2.157387 +i0.9215197
0.4	2.003675 +i1.395539	2.031403 +i1.344782	2.057207 +i1.284533	2.081835 +i1.212594	2.106196 +i1.125364	2.131584 +i1.016472
0.5	1.974508 +i1.508161	2.004238 +i1.454774	2.031443 +i1.392467	2.056875 +i1.319268	2.081397 +i1.232026	2.106196 +i1.125364

4. NUMERICAL RESULTS AND DISCUSSION

In Fig. 2, the dominant exponential decay rate (the real part $R(\gamma)$, where γ is the root of (3.5) with smallest real part) is plotted as a function of E_2/E_1 for various values of the volume fraction f. Figure 2 concerns the case where the Young's modulus of the face material is lower than that of the core material and so $E_2/E_1 > 1$. It is seen that the values of $R(\gamma)$ lie between about 0.5 to 1.5 times the value 2.106 for the homogeneous strip.

For small values of f, the pertinent roots of (3.5) are complex for the range of values of E_2/E_1 shown in Fig. 2. Their real parts increase with increasing E_2/E_1 as shown in Fig. 2. For larger values of f however, these roots may be complex (for the lower range of E_2/E_1) or real. In the latter situation, the values *decrease* with increasing E_2/E_1. On using $(3.4)_1$, the asymptotic values of these real roots as this ratio tends to infinity can be shown to coincide with the values of γ_1 given in Table 1 (when $\nu = 0.30$).

Results for the case when $E_1/E_2 > 1$ are shown in Fig. 3. For the higher range of E_1/E_2, there always exist real roots at every volume fraction f. Furthermore, these roots show a trend to decrease to zero as this ratio increases. In the limiting Case B discussed in Section 3, such roots were not anticipated. An asymptotic form for these roots may be obtained from $(3.5)_2$, under the hypotheses $E_1/E_2 \gg 1$, $\gamma \ll 1$. Using appropriate Taylor expansions, it can be shown that (see Ref. [18])

$$\gamma \sim \left[\frac{2(f^2 - 3f + 3)(1 - \nu_1^2)E_2}{f^3(1-f)(1+\nu_2)E_1}\right]^{1/2}, \tag{4.1}$$

and so we obtain a slow decay rate as suggested by Fig. 3. We observe that asymptotic estimates similar to (4.1) have been obtained in [1–3, 12] for the case of highly anisotropic materials.

It is of interest to compare the results obtained here with the experimental work of Alwar[14] when $E_1/E_2 > 1$. If we assume that $E_1/E_2 = 3600$ as in [14], and let $\nu_1 = \nu_2 = 0.3$, $f = 0.8$, then from (4.1), we obtain the estimate

$$\gamma \simeq 0.07. \tag{4.2}$$

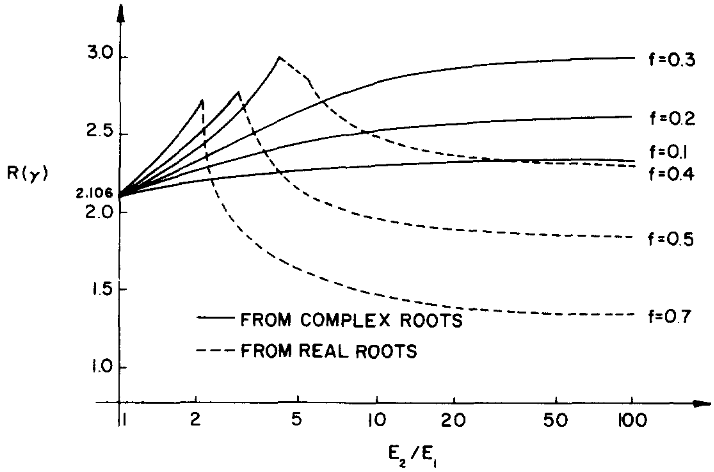

Fig. 2. Real part of first root of eqn (3.5) for plane strain at various volume fractions f. ($E_2 > E_1$, $\nu_1 = \nu_2 = 0.3$). Graph plotted on a semi-log scale.

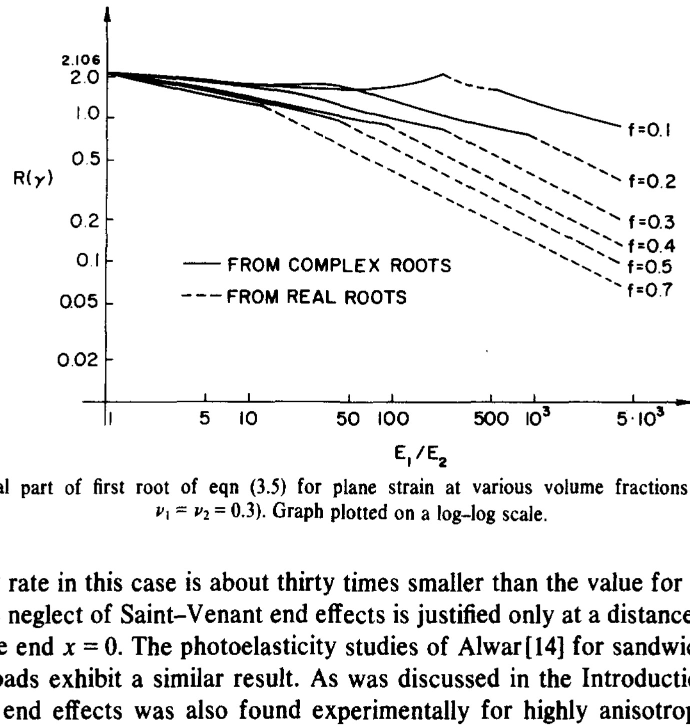

Fig. 3. Real part of first root of eqn (3.5) for plane strain at various volume fractions f. ($E_2 < E_1$, $\nu_1 = \nu_2 = 0.3$). Graph plotted on a log–log scale.

Thus the decay rate in this case is about thirty times smaller than the value for a homogeneous strip and so the neglect of Saint-Venant end effects is justified only at a distance of about thirty widths from the end $x = 0$. The photoelasticity studies of Alwar[14] for sandwich beams under concentrated loads exhibit a similar result. As was discussed in the Introduction here, such a slow decay of end effects was also found experimentally for highly anisotropic materials in [9–11], confirming earlier theoretical predictions[1–3]. The consequent lack of justification for the classic application of Saint-Venant's principle in these situations has obvious implication for practical stress analysis.

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APPENDIX

For composite material problems involving two isotropic phases, as is the case here, it has been shown by Dundurs (see [20, 21]) that the stress dependence on material properties can be conveniently described in terms of the dimensionless parameters α, β given by

$$\alpha = \frac{\Gamma(\kappa_1+1)-(\kappa_2+1)}{\Gamma(\kappa_1+1)+\kappa_2+1}, \qquad \beta = \frac{\Gamma(\kappa_1-1)-(\kappa_2-1)}{\Gamma(\kappa_1+1)+\kappa_2+1}, \qquad \Gamma = \frac{G_2}{G_1}, \tag{A1}$$

where G_i ($i = 1, 2$) are the shear moduli and

$$\begin{aligned} \kappa &= 3-4\nu && \text{(Plane Strain)} \\ &= (3-\nu)/(1+\nu) && \text{(Generalized Plane Stress).} \end{aligned} \tag{A2}$$

From (2.8), (2.16) and (2.19), it can be readily shown that the material constants E', K_1 and K_2 used in this paper may be expressed in terms of the Dundurs' constants as

$$E' = \frac{1-\alpha}{1+\alpha}, \qquad K_1 = \frac{4(\alpha-\beta)}{1+\alpha}, \qquad K_2 = \frac{-4(\alpha+\beta)}{1+\alpha}. \tag{A3}$$

While the introduction of the constants α, β was not necessary for our purposes here, the employment of these parameters has been shown to offer considerable simplification in analysis of two phase composites.